$$(x - a_1)^n + \binom{n}{2}(a_2 - a_1^2)x^{n-2} + \varphi(x) = 0,$$

where $\varphi(x)$ is a polynomial of degree $n - 3$ at most. Suppose $a_2 = a_1^2$, and write $y = x - a_1$. The equation becomes

$$y^n + \varphi(y + a_1) \equiv y^n + c_3y^{n-3} + \cdots + c_n = 0.$$

If all the c's are zero, the theorem is true. If not, let c_k be the first coefficient not zero, and c_e the last. The equation is then

$$y^n + c_ky^{n-k} + \cdots + c_ey^{n-l} = 0.$$

Apply Descartes's rule to this equation. If we call the left hand side $f(y)$, $f(y)$ and $f(-y)$ can have together at most $n - k + 1 - l$ variations of sign (k odd), or $n - k + 2 - l$ (k even), hence at most $n - 2 - l$. The equation has, therefore, at least $2 + l$ zero or complex roots. Exactly l roots are zero, however, hence it must have 2 complex. This contradicts our hypothesis that the roots were all real. Hence all the c's are zero and $f(y) \equiv y^n$, and the equation in x is $(x - a_1)^n = 0$.

Also solved by LAENAS G. WELD.

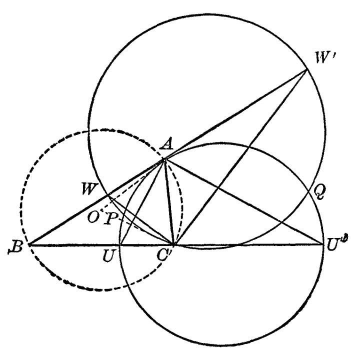
GEOMETRY.

456. Proposed by J. W. CLAWSON, Ursinus College.

The interior and exterior bisectors of the angles A, B, C of a triangle meet the opposite sides in U, U'; V, V'; W, W' respectively. Circles are drawn on UU', VV', WW' as diameters (Circles of Apollonius). Prove that (1) these three circles have a common chord. (2) The center of the circumcircle lies on this common chord.

SOLUTION BY PROPOSER.

1. $A(BC, UU')$ is a harmonic pencil; so (BC, UU') is a harmonic range. If T be any point on the circle having UU' as diameter, $T(BC, UU')$ is a harmonic



pencil. But UTU' is a right angle. Therefore TU bisects $\angle BTC$. Hence,

$$BT : TC = BU : UC = BA : AC.$$

In particular, if the circles having UU' and WW' as diameters intersect at the points P and Q,

$$BP : PC = BA : AC \text{ and } BQ : QC = BA : AC.$$

Similarly,

$$AP : PB = AC : CB \text{ and } AQ : QB = AC : CB.$$

Hence,

$$AP : PC = AB : BC \text{ and } AQ : QC = AB : BC.$$

It follows that P and Q are points on the circle having VV' as diameter. Hence the three circles UU', VV', WW' are coaxial.

2. Since (BC, UU') is a harmonic range, any circle passing through B and C is orthogonal to the circle having UU' as diameter. Hence, the circumcircle is orthogonal to that circle. Similarly the circumcircle is orthogonal to the circles VV' and WW'.

Hence, if O be the circumcenter, OA is a tangent to UU', OB to VV' and OC to WW'. But these tangents are equal in length.

Hence O is a point on the common chord of the circles UU', VV', WW'.

Also solved by C. E. Horne, David F. Kelley, Elijah Swift, Roger A. Johnson, and C. N. Schmall.

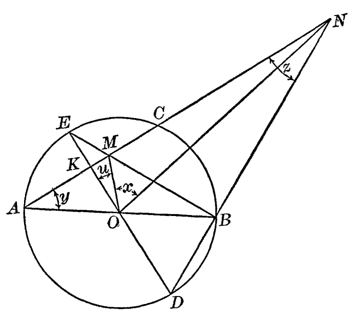
457. Proposed by NATHAN ALTSHILLER, University of Washington.

AB and AC are respectively a diameter and a chord of a circle whose center is O. The lines joining B to the extremities of the diameter perpendicular to AC, meet AC in the points M, N. Express the angle MON in terms of the angle CAB.

Solution by J. A. Caparo, University of Notre Dame.

Let E and D be the extremities of the diameter perpendicular to AC at the point K. With the notation of the figure we easily see that

$$\sin \angle AMB = \sin \angle ABN = \cos z.$$



Hence, by the sine law, letting $AB = 2R$, we have in $\triangle ANB$, $AN = 2R \cot z$, and in $\triangle AMB$, $AM = 2R \tan z$. Also from $\triangle AKO$, $AK = R\cos y$, $OK = R \sin y$. $KN = AN - AK = 2R \cot z - R \cos y$, $MK = AM - AK = 2R \tan z - R \cos y$.

$$\therefore \tan u = \frac{MK}{OK} \; \frac{2 \tan z - \cos y}{\sin y}. \quad \text{Also } \tan (x + u) = \frac{KN}{KO}.$$